

# FROM 2D TODA HIERARCHY TO CONFORMAL MAP FOR DOMAINS OF RIEMANN SPHERE

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## 1. Introduction.

In recent works [1,2,3] was found a wonderful correlation between integrable systems and meromorphic functions. They reduce a problem of effectivisation of Riemann theorem about conformal maps to calculation of a string solution of dispersionless limit of the 2D Toda hierarchy. In [4] was found a recurrent formulas for coefficients of Taylor series of the string solution. This gives, in particular, a method for calculation of the univalent conformal map from the unit disk to an arbitrary domain, described by its harmonic moments.

In the present paper we investigate some properties of these formulas. In particular, we find a sufficient condition for convergence of the Taylor series for the string solution of dispersionless limit of 2D Toda hierarchy.

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## 2. Complex domains and 2D Toda hierarchy.

In this section we remind some results of works [1, 2, 3].

We shall consider only domains generated by closed analytical contours  $\gamma$  without self-intersections on Riemann sphere  $\mathbb{C} = \mathbb{C} \cup \infty$ . The analyticity of  $\gamma$  means, that  $\gamma$  is an image of  $\gamma_0 = \{w \in \mathbb{C} | |w| = 1\}$  by a function and this function is analytical in a neighborhood of  $\gamma_0$ . A curve  $\gamma$  divides  $\mathbb{C}$  into an interior domain  $D_+$  and an external domain  $D_-$ . We shall consider, that  $D_+ \ni 0$ .

Harmonic moments of  $D_+$  are the numbers

$$v_0 = \frac{2}{\pi} \int_{D_+} \log |z| d^2 z, \quad v_k = \frac{1}{\pi} \int_{D_+} z^k d^2 z .$$

Harmonic moments of  $D_-$  are the numbers

$$t_0 = \frac{1}{\pi} \int_{D_+} d^2 z, \quad t_k = -\frac{1}{\pi k} \int_{D_-} z^{-k} d^2 z .$$

Let  $T$  be the set of analytic curves on  $\bar{\mathbb{C}}$ , parametrized by collections  $\{t_i\}$  of its harmonic moments. Consider on  $T$  the function

$$F(t) = -\frac{1}{\pi} \iint_{D_+(t)} \log \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z d^2 z' .$$

Later we consider that  $F$  is a real analytical function from  $\tilde{t} = (t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots)$ .

Consider the function

$$\varphi(z, \tilde{t}) = -\partial_{t_0} \left( \frac{1}{2} \partial_{t_0} + \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k} \right) F(\tilde{t}) ,$$

where  $\partial_t = \frac{\partial}{\partial t}$  and  $z \in \bar{\mathbb{C}}$ . Then the function  $w(z) = e^\varphi z$  is the one-sheeted meromorphic function mapping from  $D_-(t)$  to  $\{w \in \bar{\mathbb{C}} \mid |w| > 1\}$ . Therefore, if we know a Taylor series of  $F(\tilde{t})$ , we can find the functions  $p_j(t)$ , giving a representation of the function  $w(z)$  in the form

$$w(z) = \frac{1}{r} z + \sum_{j=0}^{\infty} p_j(t) z^{-j} .$$

Thus, for an effectivisation of the Riemann theorem it is sufficiently to find the Taylor series of  $F(\tilde{t})$ .

In [1 - 3] is proved that  $F(\tilde{t})$  satisfy the differential equations:

$$(z - \xi) e^{D(z)D(\xi)F} = z e^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F} , \quad (1)$$

$$(\bar{z} - \bar{\xi}) e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} = \bar{z} e^{-\partial_0 \bar{D}(\bar{z})F} - \bar{\xi} e^{-\partial_0 \bar{D}(\bar{\xi})F} , \quad (2)$$

$$1 - e^{D(z)\bar{D}(\bar{\xi})F} = \frac{1}{z\bar{\xi}} e^{-\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F} , \quad (3)$$

where  $\partial_k = \partial_{t_k}$ ,  $\bar{\partial}_k = \partial_{\bar{t}_k}$  and

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k , \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k .$$

This system of the nonlinear differential equations is well known in mathematical physics and in theory of integrable systems as dispersionless limit of the  $2D$  Toda hierarchy [6]. The solution  $F(\tilde{t})$  satisfies some additional equation, which appear in string theory and, therefore,  $F(\tilde{t})$  is called "string solution" [7]. The string solution of dispersionless limit of  $2D$  Toda hierarchy is appeared also in matrix models and in some other problems of mathematical physics. Thus a description of it has an independent interest.

### 3. Taylor series for the string solution of dispersionless limit of the 2D Toda hierarchy.

In [ 4 ] was found a representation of  $F$  in form of Taylor series

$$F = \sum N(i|i_1\dots, i_k|\bar{i}_1\dots, \bar{i}_k)t_0^i t_{i_1} \cdots t_{i_k} \bar{t}_{\bar{i}_1} \cdots \bar{t}_{\bar{i}_k} .$$

The formulas for  $N$  are found by the following scheme. At first, using some combinatorial calculations, we transform the equation (1) to an infinite system of equations

$$\begin{aligned} & \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} F = \\ & = \sum_{m=1}^{\infty} \left( \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ \ell_1+\dots+\ell_m=m+k-2 \\ s_j, \ell_j \geq 1}} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} T_{i_1 \cdots i_k} \left( \begin{matrix} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{matrix} \right) \partial_0^{\ell_1} \partial_{s_1} F \cdots \partial_0^{\ell_m} \partial_{s_m} F \right). \end{aligned} \quad (4)$$

In passing we find some recurrent formulas for calculation of  $T$ .

Later, using the definition of  $F$  as a function on the space of analytical curves, we find, that

$\partial_0 F|_{t_0} = -t_0 + t_0 \ln t_0$  and  $\partial_k F|_{t_0} = 0$ , if  $k > 0$ ,  
where here and later  $|_{t_0}$  means the restriction of a function on a straight line  $t_1 = \bar{t}_1 = t_2 = \bar{t}_2 = \cdots = 0$ .

For this formula and from the equation (3) follow, that

$$\partial_i \bar{\partial}_j F|_{t_0} = \begin{cases} 0, & \text{if } i \neq j, \\ it_0^i, & \text{if } i = j. \end{cases}$$

Later, using (4) and the symmetry of the equations (1) - (3) we find, that

$$\partial_i \bar{\partial}_{i_1} \cdots \bar{\partial}_{i_k} F|_{t_0} = \bar{\partial}_i \partial_{i_1} \cdots \partial_{i_k} F|_{t=0} = \begin{cases} 0, & \text{if } i_1 + \cdots + i_k \neq i, \\ i_1 \cdots i_k \frac{i!}{(i-k+1)!} t_0^{i-k+1}, & \text{if } i_1 + \cdots + i_k = i. \end{cases}$$

This condition and the equation (4) give some recurrent formulas for coefficients  $N$ . As the final result we get

**Theorem 1.** *In the domain of its convergence the formal series*

$$\begin{aligned} F &= \frac{1}{2} t_0^2 \log t_0 - \frac{3}{4} t_0^2 + \\ &+ \sum_{\substack{k, \bar{k}, n_r, \bar{n}_r \geq 1 \\ 0 < i_1 < \cdots < i_k \\ 0 < \bar{i}_1 < \cdots < \bar{i}_{\bar{k}} \\ i - (n_1 + \cdots + n_k + \bar{n}_1 + \cdots + \bar{n}_{\bar{k}}) + 2 \geq 0}} \frac{i_1^{n_1} \cdots i_k^{n_k}}{n_1! \cdots n_k!} \frac{\bar{i}_1^{\bar{n}_1} \cdots \bar{i}_{\bar{k}}^{\bar{n}_{\bar{k}}}}{\bar{n}_1! \cdots \bar{n}_{\bar{k}}!} N_i^2 \left( \begin{matrix} i_1, & \dots, & i_k \\ n_1, & \dots, & n_k \end{matrix} \middle| \begin{matrix} \bar{i}_1, & \dots, & \bar{i}_{\bar{k}} \\ \bar{n}_1, & \dots, & \bar{n}_{\bar{k}} \end{matrix} \right) \times \end{aligned}$$

$$\times t_0^{i-(n_1+\dots+n_k+\bar{n}_1+\dots+\bar{n}_k)+2} t_{i_1}^{n_1} \dots t_{i_k}^{n_k} \bar{t}_{i_1}^{\bar{n}_1} \dots \bar{t}_{i_k}^{\bar{n}_k}$$

is the string solution of dispersionless limit of 2D Toda hierarchy. In this formula the coefficients  $N^2$  are found by follow recurrent rules.

$P_{i,j}(s_1, \dots, s_m) = \# \{(i_1, \dots, i_m) \mid i = i_1 + \dots + i_m, 1 \leq i_r \leq s_r - 1\}$ , where  $\#Q$  is the cardinality of the set  $Q$ ;

$$T_{i,j}^1(s_1, \dots, s_m) =$$

$$\sum_{\substack{k \geq 1 \\ n_1 + \dots + n_k = m \\ n_r \geq 1}} \frac{1}{kn_1! \dots n_k!} P_{i,j} \left( \underbrace{s_1 + \dots + s_{n_1}}_{n_1}, \dots, \underbrace{s_{n_1 + \dots + n_{k-1} + 1} + \dots + s_{n_1 + \dots + n_k}}_{n_k} \right);$$

$$T_{i_1, i_2}^2 \begin{pmatrix} s_1, & \dots, & s_m \\ 1, & \dots, & 1 \end{pmatrix} = T_{i_1, i_2}^1(s_1, \dots, s_m);$$

$$T_{i_1, \dots, i_k}^2 \begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{pmatrix} =$$

$$= \sum_{\substack{1 \leq i \leq j \leq m \\ s, l \geq 1}} l T_{s, i_k}^1(s_i, \dots, s_j) T_{i_1, \dots, i_{k-1}}^2 \begin{pmatrix} s_1, & \dots, & s_{i-1}, & s, & s_{j+1}, & \dots, & s_m \\ l_1, & \dots, & l_{i-1}, & l, & l_{j+1}, & \dots, & l_m \end{pmatrix},$$

where  $s = s_i + \dots + s_j - i_k$ ,  $l = (l_i - 1) + \dots + (l_j - 1)$ ;

$$S_{\bar{i}_1, \dots, \bar{i}_k} \begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{pmatrix} =$$

$$= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \{\bar{i}_1, \dots, \bar{i}_k\} \\ \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} = s_r \\ s_r - n_r - \bar{l}_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - \bar{l}_1 + 1)!(\bar{l}_1 - 1)!} \times \dots \times \frac{(s_m - 1)!}{(s_m - n_m - \bar{l}_m + 1)!(\bar{l}_m - 1)!};$$

$$N_i^1(i_1, \dots, i_k | \bar{i}_1, \dots, \bar{i}_k) = 0, \quad \text{if } i \neq i_1 + \dots + i_k \text{ or } i \neq \bar{i}_1 + \dots + \bar{i}_k;$$

in the other cases

$$N_i^1(i | \bar{i}_1, \dots, \bar{i}_k) = \frac{(i-1)!}{(i-k+1)!};$$

$$N_i^1(i_1, \dots, i_k | \bar{i}) = \frac{(i-1)!}{(i-k+1)!};$$

$$N_i^1(i_1, \dots, i_k | \bar{i}_1, \dots, \bar{i}_k) =$$

$$= \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + \dots + i_k \\ l_1 + \dots + l_m = m + k - 2 \\ s_r, l_r \geq 1}} (-1)^{m+1} S_{\bar{i}_1, \dots, \bar{i}_k} \begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{pmatrix} T_{i_1, \dots, i_k}^2 \begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{pmatrix}, \quad \text{if}$$

$$k, \bar{k} > 1;$$

$$\begin{aligned}
 N_i^2 \left( \begin{array}{c|c} i_1, \dots, i_k & \bar{i}_1, \dots, \bar{i}_{\bar{k}} \\ n_1, \dots, n_k & \bar{n}_1, \dots, \bar{n}_{\bar{k}} \end{array} \right) &= \\
 &= N_i^1 \left( \underbrace{i_1, \dots, i_1}_{n_1}, \dots, \underbrace{i_k, \dots, i_k}_{n_k} \middle| \underbrace{\bar{i}_1, \dots, \bar{i}_1}_{\bar{n}_1}, \dots, \underbrace{\bar{i}_{\bar{k}}, \dots, \bar{i}_{\bar{k}}}_{\bar{n}_{\bar{k}}} \right) . \square
 \end{aligned}$$

For  $t_3 = \bar{t}_3 = t_4 = \bar{t}_4 = \dots = 0$  this theorem goes to the formula from [2]

$$F = -\frac{3}{4}t_0^2 + \frac{1}{2}t_0^2 \ln \left( \frac{t_0}{1 - 4|t_2|^2} \right) + \frac{t_0}{1 - 4|t_2|^2} (|t_1|^2 + t_1^2 \bar{t}_2 + \bar{t}_1^2 t_2) .$$

First two authors construct a computer program, calculating any coefficient  $N_i^2(\dots)$ . Calculations by this program lead to hypothesis, that all coefficients  $N_i^2(\dots)$  are nonnegative.

#### 4. Some properties of the coefficients for the series $F$ .

The combinatorial coefficients  $N_i^2(\dots)$  have some remarkable properties. For example,

**Theorem 2.**  $N_i^2 \left( \begin{array}{c|c} i_1 \cdots i_k & 1 \\ n_1 \cdots n_k & \bar{n}_1 \end{array} \right) = \begin{cases} (i-1)!, & \text{if } k = n_1 = 1, \ i = i_1 = \bar{n}_1, \\ 0 & \text{in the other cases.} \end{cases}$

*Proof.* According to our definition

$$\begin{aligned}
 & \underbrace{S_{1, \dots, 1}}_{\bar{k}} \left( \begin{array}{c|c} s_1, \dots, s_m \\ l_1, \dots, l_m \end{array} \right) = \\
 &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \underbrace{\{1, \dots, 1\}}_{\bar{k}} \\ \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} = s_r \\ s_r - n_r - \bar{\ell}_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \dots \times \\
 & \times \dots \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \delta_{\ell_1, 1} \dots \delta_{\ell_m, 1} \frac{\bar{k}!}{s_1 \dots s_m} .
 \end{aligned}$$

Thus if  $k > 2$ , then

$$\begin{aligned}
 & N_i^1(i_1, \dots, i_k | 1, \dots, 1) = \\
 &= \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + \dots + i_k \\ l_1 + \dots + l_m = m + k - 2 \\ s_r, l_r \geq 1}} (-1)^{m+1} S_{1, \dots, 1} \left( \begin{array}{c|c} s_1, \dots, s_m \\ l_1, \dots, l_m \end{array} \right) \times \\
 & \times T_{i_1, \dots, i_k}^2 \left( \begin{array}{c|c} s_1, \dots, s_m \\ l_1, \dots, l_m \end{array} \right) = 0 .
 \end{aligned}$$

Let now  $k = 2$  ( $i_1, i_2 \geq 1$ ). Then

$$\begin{aligned}
& N_i^1(i_1, i_2 | \underbrace{1, \dots, 1}_{\bar{k}}) = \\
& = \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + i_2 \\ s_r \geq 1}} (-1)^{m+1} \underbrace{S_{1, \dots, 1}}_{\bar{k}} \left( \begin{matrix} s_1, & \dots, & s_m \\ 1, & \dots, & 1 \end{matrix} \right) \times \\
& \quad \times T_{i_1, i_2}^2 \left( \begin{matrix} s_1, & \dots, & s_m \\ 1, & \dots, & 1 \end{matrix} \right) = \\
& = \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + i_2 \\ s_r \geq 1}} (-1)^{m+1} \frac{\bar{k}!}{s_1 \dots s_m} T_{i_1, i_2}^2 \left( \begin{matrix} s_1, & \dots, & s_m \\ 1, & \dots, & 1 \end{matrix} \right) = \\
& = \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + i_2 \\ s_r \geq 1}} (-1)^{m+1} \frac{\bar{k}!}{s_1 \dots s_m} T_{i_1, i_2}^1(s_1, \dots, s_m) = \\
& = \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + i_2 \\ s_r \geq 1}} (-1)^{m+1} \frac{\bar{k}!}{s_1 \dots s_m} \sum_{\substack{k \geq 1 \\ n_1 + \dots + n_k = m \\ n_r \geq 1}} \frac{1}{kn_1! \dots n_k!} \times \\
& \quad \times P_{i, j} \left( \underbrace{s_1 + \dots + s_{n_1}}_{n_1}, \dots, \underbrace{s_{n_1 + \dots + n_{k-1} + 1} + \dots + s_{n_1 + \dots + n_k}}_{n_k} \right) = \\
& = \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + i_2 \\ k \geq 1 \\ n_1 + \dots + n_k = m}} \frac{(-1)^{m+1}}{kn_1! \dots n_k!} \frac{\bar{k}!}{s_1 \dots s_m} \times \\
& \quad \times P_{i, j} \left( \underbrace{s_1 + \dots + s_{n_1}}_{n_1}, \dots, \underbrace{s_{n_1 + \dots + n_{k-1} + 1} + \dots + s_{n_1 + \dots + n_k}}_{n_k} \right) = \\
& = \sum_{\substack{k \geq 1 \\ \tilde{s}_1 + \dots + \tilde{s}_k = i_1 + i_2}} P_{i, j}(\tilde{s}_1, \dots, \tilde{s}_k) \times
\end{aligned}$$

$$\begin{aligned}
 & \times \sum_{\substack{n_r \geq 1, n_1 + \dots + n_k = m \\ \underbrace{s_{n_1 + \dots + n_{r-1} + 1} + \dots + s_{n_1 + \dots + n_r} = \tilde{s}_r}_{n_r}}} \frac{(-1)^{m+1} \bar{k}!}{k n_1! \dots n_k! s_1 \dots s_m} = \\
 & = \sum_{\substack{k \geq 1 \\ \tilde{s}_1 + \dots + \tilde{s}_k = i_1 + i_2}} -\frac{\bar{k}!}{k} P_{i,j}(\tilde{s}_1, \dots, \tilde{s}_k) \times \\
 & \times \prod_{1 \leq r \leq k} \sum_{\substack{n_r \geq 1 \\ s_1 + \dots + s_{n_r} = \tilde{s}_r}} \frac{(-1)^{n_r}}{n_r! s_1 \dots s_{n_r}}.
 \end{aligned}$$

In addition if  $s > 1$ , then

$$\begin{aligned}
 \sum_{\substack{n \geq 1 \\ s_1 + \dots + s_n = s}} \frac{(-1)^n}{n! s_1 \dots s_n} &= \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(-1)^n}{n!} \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right)^n \Big|_{x=0} = \\
 &= \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(-1)^n}{n!} (-\log(1-x))^n \Big|_{x=0} = \\
 &= \frac{1}{s!} \frac{\partial^s}{\partial x^s} \sum_{n \geq 1} \frac{(\log(1-x))^n}{n!} \Big|_{x=0} = \frac{1}{s!} \frac{\partial^s}{\partial x^s} (\exp(\log(1-x)) - 1) \Big|_{x=0} = 0.
 \end{aligned}$$

Thus

$$\begin{aligned}
 N_i^1(i_1, i_2 | \underbrace{1, \dots, 1}_{\bar{k}}) &= \sum_{\substack{k \geq 1 \\ \tilde{s}_1 + \dots + \tilde{s}_k = i_1 + i_2}} -\frac{\bar{k}!}{k} P_{i,j}(\tilde{s}_1, \dots, \tilde{s}_k) \times \\
 & \times \prod_{1 \leq r \leq k} \sum_{\substack{n_r \geq 1 \\ s_1 + \dots + s_{n_r} = \tilde{s}_r}} \frac{(-1)^{n_r}}{n_r! s_1 \dots s_{n_r}} = 0,
 \end{aligned}$$

because  $P_{i,j}(1, \dots, 1) = 0$ .

If  $k = 1$ , then from our definition it follows that

$$N_i^2 \left( \begin{array}{c} i_1 \\ n_1 \end{array} \middle| \begin{array}{c} 1 \\ \bar{n}_1 \end{array} \right) = \begin{cases} (i-1)!, & \text{if } n_1 = 1, i = i_1 = \bar{n}_1, \\ 0 & \text{in the other cases.} \end{cases} \quad \square$$

## 5. Convergence conditions for the Taylor series.

The recurrent formulas for coefficients of the Taylor series  $F$  give possibility to estimate the coefficients and to find sufficient convergence conditions for  $F$ .

**Theorem 3.** Let  $\tilde{t} = (t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots)$  be such that  $t_i, \bar{t}_i = 0$  for  $i > n, 0 < t_0 < 1$  and  $|t_i|, |\bar{t}_i| \leq (4n^3 2^n e^n)^{-1}$ . Then the series  $F(\tilde{t})$  is convergence.

A proof is based on a sequential of estimations of all values, utilized in the definition of  $N^2$ . Present these estimations:

1. Let  $i + j = s_1 + \dots + s_m$ . Then  $P_{ij}(s_1, \dots, s_m) \leq \min(C_{i-1}^{m-1}, C_{j-1}^{m-1})$ .

*Proof.*

$$\begin{aligned} P_{j,i}(s_1, \dots, s_m) &= P_{i,j}(s_1, \dots, s_m) = \#\{(i_1, \dots, i_m) \mid i = \\ &= i_1 + \dots + i_m, 1 \leq i_r \leq s_r - 1\} \leq \\ &\leq \#\{(i_1, \dots, i_m) \mid i = i_1 + \dots + i_m, 1 \leq i_r\} = C_{i-1}^{m-1}. \quad \square \end{aligned}$$

2. Let  $i + j = s_1 + \dots + s_m$ . Then  $T_{ij}^1(s_1, \dots, s_m) \leq \frac{\ell^{m-1}}{m!}$ , where  $\ell = \min(i, j)$ .

*Proof.*

$$\begin{aligned} T_{j,i}^1(s_1, \dots, s_m) &= T_{i,j}^1(s_1, \dots, s_m) = \sum_{\substack{k \geq 1 \\ n_1 + \dots + n_k = m \\ n_r \geq 1}} \frac{1}{kn_1! \dots n_k!} \times \\ &\times P_{i,j} \left( \underbrace{s_1 + \dots + s_{n_1}}_{n_1}, \dots, \underbrace{s_{n_1 + \dots + n_{k-1} + 1} + \dots + s_{n_1 + \dots + n_k}}_{n_k} \right) \leq \\ &\leq \sum_{\substack{k \geq 1 \\ n_1 + \dots + n_k = m \\ n_r \geq 1}} \frac{C_{i-1}^{k-1}}{kn_1! \dots n_k!} = \\ &= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geq 1} C_i^k \sum_{n_1 + \dots + n_k = m} \frac{1}{n_1! \dots n_k!} = \\ &= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \sum_{k \geq 1} C_i^k \left( x + \frac{x^2}{2!} + \dots \right) \Big|_{x=0} = \\ &= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} \left( \sum_{k \geq 1} C_i^k (e^x - 1)^k \right) \Big|_{x=0} = \\ &= \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} ((1 + (e^x - 1))^i - 1) \Big|_{x=0} = \frac{1}{m!} \frac{\partial^m}{\partial x^m} \frac{1}{i} (e^{ix} - 1) \Big|_{x=0} = \frac{i^{m-1}}{m!}. \quad \square \end{aligned}$$

3. Let  $i_1 + \dots + i_k = s_1 + \dots + s_m$  and  $(\ell_1 - 1) + \dots + (\ell_m - 1) = k - 2$ . Then  $T_{i_1, \dots, i_k}^2 \left( \begin{smallmatrix} s_1 \dots s_m \\ \ell_1 \dots \ell_m \end{smallmatrix} \right) \leq \frac{I^{m-1} (k-1)^m (k-2)!}{m!}$ , where  $I = \max(i_r)$ .



*Proof.* We use an induction by  $k$ . If  $k = 2$ , then

$$\begin{aligned} T_{i_1, i_2}^2 \begin{pmatrix} s_1, & \dots, & s_m \\ 1, & \dots, & 1 \end{pmatrix} &= T_{i_1, i_2}^1(s_1, \dots, s_m) \leq \\ &\leq \frac{I^{m-1}}{m!} = \frac{I^{m-1}(k-1)^m(k-2)!}{m!} \Big|_{k=2}. \end{aligned}$$

Let  $k > 2$ . Note at first that, if  $l = (l_i - 1) + \dots + (l_j - 1)$ , then

$$\begin{aligned} \sum_{1 \leq i \leq j \leq m, j-i=d} \frac{l}{(k-2)(d+1)} &= \frac{\sum_{1 \leq i \leq j \leq m, j-i=d} (l_i - 1) + \dots + (l_j - 1)}{(k-2)(d+1)} \leq \\ &\leq \frac{\sum_{1 \leq t \leq m} (l_t - 1)(d+1)}{(k-2)(d+1)} = 1. \end{aligned}$$

Thus

$$\begin{aligned} T_{i_1, \dots, i_k}^2 \begin{pmatrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{pmatrix} &= \sum_{\substack{1 \leq i \leq j \leq m \\ s, l \geq 1}} l T_{s, i_k}^1(s_i, \dots, s_j) \times \\ &\times T_{i_1, \dots, i_{k-1}}^2 \begin{pmatrix} s_1, & \dots, & s_{i-1}, & s, & s_{j+1}, & \dots, & s_m \\ l_1, & \dots, & l_{i-1}, & l, & l_{j+1}, & \dots, & l_m \end{pmatrix} \leq \\ &\leq \sum_{1 \leq i \leq j \leq m} l \frac{I^{j-i}}{(j-i+1)!} \frac{I^{m-(j-i+1)+1-1}(k-2)^{m-(j-i+1)+1}(k-3)!}{(m-(j-i+1)+1)!} = \\ &= I^{m-1}(k-3)! \sum_{1 \leq i \leq j \leq m} l \frac{(k-2)^{m-(j-i)}}{(j-i+1)!(m-(j-i))!} = \\ &= \frac{I^{m-1}(k-2)!}{m!} \sum_{1 \leq i \leq j \leq m, d=j-i} \frac{l}{(k-2)(d+1)} \frac{m!}{d!(m-d)!} (k-2)^{m-d} = \\ &= \frac{I^{m-1}(k-2)!}{m!} \sum_{d=0}^{m-1} \frac{m!}{d!(m-d)!} (k-2)^{m-d} \left( \sum_{1 \leq i \leq j \leq m, j-i=d} \frac{l}{(k-2)(d+1)} \right) \leq \\ &\leq \frac{I^{m-1}(k-2)!}{m!} \sum_{d=0}^{m-1} \frac{m!}{d!(m-d)!} (k-2)^{m-d} \leq \frac{I^{m-1}(k-2)!}{m!} \sum_{t=0, t=m-d}^m C_m^t (k-2)^t = \\ &= \frac{I^{m-1}(k-2)!}{m!} ((k-2)+1)^m = \frac{I^{m-1}(k-1)^m(k-2)!}{m!}. \quad \square \end{aligned}$$

4. Let  $\bar{I} = \max_r(\bar{i}_r)$  and

$$\begin{aligned} \tilde{S}_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}}(m, k) &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ (l_1-1) + \dots + (l_m-1) = k-2 \\ n_r, l_r \geq 1 \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \\ s_r - n_r - l_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \dots \times \\ &\times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!}. \end{aligned}$$

Then  $\tilde{S}_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}}(m, k) \leq m(\bar{k} - 1)! C_{\bar{I}\bar{k}-\bar{k}}^{k-2} C_{\bar{I}\bar{k}}^{\bar{k}-m}$ .

*Proof.* We use the equality

$$\begin{aligned} \sum_{\tilde{n}_1 + \dots + \tilde{n}_m = \bar{k} - m} C_{\bar{I}\tilde{n}_1 + \bar{I}}^{\tilde{n}_1} \times \dots \times C_{\bar{I}\tilde{n}_m + \bar{I}}^{\tilde{n}_m} \frac{\bar{I}}{\bar{I}\tilde{n}_1 + \bar{I}} \times \dots \times \frac{\bar{I}}{\bar{I}\tilde{n}_m + \bar{I}} &= \\ &= C_{\bar{I}(\bar{k}-m) + m\bar{I}}^{\bar{k}-m} \frac{m\bar{I}}{\bar{I}(\bar{k}-m) + m\bar{I}} \end{aligned}$$

that follows from [8,5.63]. Then

$$\begin{aligned} \tilde{S}_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}}(m, k) &= \\ &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ (l_1-1) + \dots + (l_m-1) = k-2 \\ n_r, l_r \geq 1 \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \\ s_r - n_r - l_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \dots \times \\ &\times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \\ &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \geq n_r \geq 1}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \dots \times \frac{(s_m - 1)!}{(s_m - n_m)!} \times \\ &\times \sum_{\substack{(l_1-1) + \dots + (l_m-1) = k-2 \\ s_r - n_r - l_r + 1 \geq 0}} \frac{(s_1 - n_1)!}{(s_1 - n_1 - l_1 + 1)!(l_1 - 1)!} \times \dots \times \\ &\times \frac{(s_m - n_m)!}{(s_m - n_m - l_m + 1)!(l_m - 1)!} = \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \geq n_r \geq 1}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \dots \times \\
 &\times \frac{(s_m - 1)!}{(s_m - n_m)!} \sum_{\bar{l}_1 + \dots + \bar{l}_m = k-2} C_{s_1 - n_1}^{\bar{l}_1} \dots C_{s_m - n_m}^{\bar{l}_m} = \\
 &= \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \geq n_r \geq 1}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \dots \times \\
 &\times \frac{(s_m - 1)!}{(s_m - n_m)!} C_{(s_1 + \dots + s_m) - (n_1 + \dots + n_m)}^{k-2} = \\
 &= C_{\bar{i}_1 + \dots + \bar{i}_{\bar{k}} - \bar{k}}^{k-2} \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \geq n_r \geq 1}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \dots \times \frac{(s_m - 1)!}{(s_m - n_m)!} \leq \\
 &\leq \underbrace{\tilde{S}_{\bar{I}, \dots, \bar{I}}}_{\bar{k}}(m, k) = C_{\bar{I}\bar{k} - \bar{k}}^{k-2} \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \\ \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{I}, \dots, \bar{I}\}, n_r \geq 1 \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} = In_r}} \frac{(s_1 - 1)!}{(s_1 - n_1)!} \times \dots \times \frac{(s_m - 1)!}{(s_m - n_m)!} = \\
 &= C_{\bar{I}\bar{k} - \bar{k}}^{k-2} \sum_{\substack{n_1 + \dots + n_m = \bar{k} \\ n_r \geq 1}} \frac{\bar{k}!}{n_1! \dots n_m!} \frac{(\bar{I}n_1 - 1)!}{(\bar{I}n_1 - n_1)!} \times \dots \times \frac{(\bar{I}n_m - 1)!}{(\bar{I}n_m - n_m)!} = \\
 &= \bar{k}! C_{\bar{I}\bar{k} - \bar{k}}^{k-2} \sum_{\substack{\bar{n}_1 + \dots + \bar{n}_m = \bar{k} - m \\ \bar{n}_r = n_r - 1 \geq 0}} \frac{(\bar{I}\bar{n}_1 + \bar{I} - 1)!}{(\bar{n}_1 + 1)! (\bar{I}\bar{n}_1 + \bar{I} - \bar{n}_1 - 1)!} \times \dots \times \\
 &\quad \times \frac{(\bar{I}\bar{n}_m + \bar{I} - 1)!}{(\bar{n}_m + 1)! (\bar{I}\bar{n}_m + \bar{I} - \bar{n}_m - 1)!} = \\
 &= \bar{k}! C_{\bar{I}\bar{k} - \bar{k}}^{k-2} \sum_{\bar{n}_1 + \dots + \bar{n}_m = \bar{k} - m} \frac{\bar{I}\bar{n}_1 + \bar{I} - \bar{n}_1}{(\bar{I}\bar{n}_1 + \bar{I})(\bar{n}_1 + 1)} \frac{(\bar{I}\bar{n}_1 + \bar{I})!}{\bar{n}_1! (\bar{I}\bar{n}_1 + \bar{I} - \bar{n}_1)!} \times \dots \times \\
 &\quad \times \frac{\bar{I}\bar{n}_m + \bar{I} - \bar{n}_m}{(\bar{I}\bar{n}_m + \bar{I})(\bar{n}_m + 1)} \frac{(\bar{I}\bar{n}_m + \bar{I})!}{\bar{n}_m! (\bar{I}\bar{n}_m + \bar{I} - \bar{n}_m)!} \leq \\
 &\leq \bar{k}! C_{\bar{I}\bar{k} - \bar{k}}^{k-2} \sum_{\bar{n}_1 + \dots + \bar{n}_m = \bar{k} - m} \frac{\bar{I}}{\bar{I}\bar{n}_1 + \bar{I}} C_{\bar{I}\bar{n}_1 + \bar{I}}^{\bar{n}_1} \times \dots \times \frac{\bar{I}}{\bar{I}\bar{n}_m + \bar{I}} C_{\bar{I}\bar{n}_m + \bar{I}}^{\bar{n}_m} =
 \end{aligned}$$

$$= \bar{k}! C_{\bar{k}-\bar{k}}^{k-2} C_{\bar{I}(\bar{k}-m)+m\bar{I}}^{\bar{k}-m} \frac{m\bar{I}}{\bar{I}(\bar{k}-m)+m\bar{I}} \leq m(\bar{k}-1)! C_{\bar{k}-\bar{k}}^{k-2} C_{\bar{I}\bar{k}}^{\bar{k}-m}. \quad \square$$

$$5. N_i^1(i_1, \dots, i_k | \bar{i}_1, \dots, \bar{i}_{\bar{k}}) \leq (k-1)! (\bar{k}-1)! e^{I(k-1)} 2^{\bar{I}\bar{k}-\bar{k}} 2^{\bar{I}\bar{k}}.$$

*Proof.*

$$\begin{aligned} & N_i^1(i_1, \dots, i_k | \bar{i}_1, \dots, \bar{i}_{\bar{k}}) = \\ &= \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + \dots + i_k \\ l_1 + \dots + l_m = m + k - 2 \\ s_r, l_r \geq 1}} (-1)^{m+1} S_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}} \left( \begin{matrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{matrix} \right) \times \\ & \quad \times T_{i_1, \dots, i_k}^2 \left( \begin{matrix} s_1, & \dots, & s_m \\ l_1, & \dots, & l_m \end{matrix} \right) \leq \\ & \leq \sum_{\substack{m \geq 1 \\ s_1 + \dots + s_m = i_1 + \dots + i_k \\ l_1 + \dots + l_m = m + k - 2 \\ s_r, l_r \geq 1}} \frac{I^{m-1} (k-1)^m (k-2)!}{m!} \times \\ & \times \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} = s_r \\ s_r - n_r - l_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)! (l_1 - 1)!} \times \dots \times \\ & \quad \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)! (l_m - 1)!} = \\ & = \sum_{m \geq 1} \frac{I^{m-1} (k-1)^{m-1} (k-2)!}{m!} \times \\ & \times \sum_{\substack{\{\bar{i}_1^1, \dots, \bar{i}_1^{n_1}\} \sqcup \dots \sqcup \{\bar{i}_m^1, \dots, \bar{i}_m^{n_m}\} = \\ = \{\bar{i}_1, \dots, \bar{i}_{\bar{k}}\} \\ (l_1 - 1) + \dots + (l_m - 1) = k - 2 \\ n_r, l_r \geq 1 \\ s_r = \bar{i}_r^1 + \dots + \bar{i}_r^{n_r} \\ s_r - n_r - l_r + 1 \geq 0}} \frac{(s_1 - 1)!}{(s_1 - n_1 - l_1 + 1)! (l_1 - 1)!} \times \dots \times \\ & \quad \times \frac{(s_m - 1)!}{(s_m - n_m - l_m + 1)! (l_m - 1)!} = \\ & = \sum_{m \geq 1} \frac{I^{m-1} (k-1)^{m-1} (k-2)!}{m!} \tilde{S}_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}}(m, k) \leq \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{m \geq 1} \frac{I^{m-1}(k-1)^m(k-2)!}{m!} \tilde{S}_{\bar{i}_1, \dots, \bar{i}_{\bar{k}}}(m, k) \leq \\
 &\leq \sum_{m \geq 1} \frac{I^{m-1}(k-1)^m(k-2)!}{m!} m(\bar{k}-1)! C_{\bar{I}\bar{k}-\bar{k}}^{k-2} C_{\bar{I}\bar{k}}^{\bar{k}-m} = \\
 &= (k-1)!(\bar{k}-1)! \sum_{m \geq 1} \frac{I^{m-1}(k-1)^{m-1}}{(m-1)!} C_{\bar{I}\bar{k}-\bar{k}}^{k-2} C_{\bar{I}\bar{k}}^{\bar{k}-m} \leq \\
 &\leq (k-1)!(\bar{k}-1)! e^{I(k-1)} 2^{\bar{I}\bar{k}-\bar{k}} 2^{\bar{I}\bar{k}}. \quad \square
 \end{aligned}$$

*Proof of theorem 3.* The coefficient for  $t_0 \dots t_{i_1}^{n_1} \dots t_{i_I}^{n_I} \bar{t}_{\bar{i}_1}^{\bar{n}_1} \dots \bar{t}_{\bar{i}_{\bar{I}}}^{\bar{n}_{\bar{I}}}$  is equal

$$\begin{aligned}
 &\frac{i_1^{n_1} \dots i_I^{n_I}}{n_1! \dots n_I!} \frac{\bar{i}_1^{\bar{n}_1} \dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1! \dots \bar{n}_{\bar{I}}!} N_i^2 \left( \begin{array}{c|c} i_1, \dots, i_I & \bar{i}_1, \dots, \bar{i}_{\bar{I}} \\ n_1, \dots, n_I & \bar{n}_1, \dots, \bar{n}_{\bar{I}} \end{array} \right) = \\
 &= \frac{i_1^{n_1} \dots i_I^{n_I}}{n_1! \dots n_I!} \frac{\bar{i}_1^{\bar{n}_1} \dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1! \dots \bar{n}_{\bar{I}}!} N_i^1 \left( \underbrace{i_1, \dots, i_1}_{n_1}, \dots, \underbrace{i_I, \dots, i_I}_{n_I} \middle| \underbrace{\bar{i}_1, \dots, \bar{i}_1}_{\bar{n}_1}, \dots, \underbrace{\bar{i}_{\bar{k}}, \dots, \bar{i}_{\bar{I}}}_{\bar{n}_{\bar{I}}} \right) \leq \\
 &\leq \frac{i_1^{n_1} \dots i_I^{n_I}}{n_1! \dots n_I!} \frac{\bar{i}_1^{\bar{n}_1} \dots \bar{i}_{\bar{I}}^{\bar{n}_{\bar{I}}}}{\bar{n}_1! \dots \bar{n}_{\bar{I}}!} k! \bar{k}! e^{\bar{I}(k-1)} 2^{2\bar{I}\bar{k}} \leq \tilde{I}^K e^{\bar{I}K} 2^{\bar{I}K} \frac{k! \bar{k}!}{n_1! \dots n_I! \bar{n}_1! \dots \bar{n}_{\bar{I}}!} \leq \\
 &\leq \tilde{I}^K e^{\bar{I}K} 2^{\bar{I}K} \tilde{I}^K \leq (\tilde{I}^2 2^{\bar{I}} e^{\bar{I}})^K,
 \end{aligned}$$

where  $k = n_1 + \dots + n_I$ ,  $\bar{k} = \bar{n}_1 + \dots + \bar{n}_{\bar{I}}$ ,  $K = k + \bar{k}$  and  $\tilde{I} = \max(I, \bar{I})$ .

Consider now monomials from  $t_0, t_1, \bar{t}_1, \dots, t_n, \bar{t}_n$  of degree  $K$ . The number of such monomial is  $(2n)^K$ . Thus its sum in the series is not more that

$$(n^2 2^n e^n)^K (2n)^K (4n^3 2^n e^n)^{-K} \leq 2^{-K}.$$

This implies the convergence of the series  $F(\tilde{t})$ .  $\square$

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